

Turán's theorem inverted

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Abstract

Let $K_r^+(s_1, \dots, s_r)$ be the complete r -partite graph with parts of size $s_1 \geq 2, s_2, \dots, s_r$ with an edge added to the first part. Letting $t_r(n)$ be the number of edges of the r -partite Turán graph of order n , we prove that:

(A) For all $r \geq 2$ and all sufficiently small $\varepsilon > 0$, every graph of sufficiently large order n with $t_r(n) + 1$ edges contains a $K_r^+(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$.

(B) For all $r \geq 2$, there exists $c > 0$ such that every graph of sufficiently large order n with $t_r(n) + 1$ edges contains a $K_r^+(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor)$.

These assertions extend results of Erdős from 1963.

We also give corresponding stability results

Keywords: *clique; r -partite graph; stability, Turán's theorem*

1 Introduction

This note is part of an ongoing project aiming to renovate some classical results in extremal graph theory, see, e.g., [3], [7, 10].

Let $t_r(n)$ be the number of edges of the r -partite Turán graph of order n . Turán's theorem implies that every graph on n vertices with $t_r(n) + 1$ edges contains a K_{r+1} , the complete graph of order $r + 1$. Thus, it is natural to ask:

Which supergraphs of K_{r+1} are present in graphs on n vertices with $t_r(n) + 1$ edges?

A partial answer to this question was stated by Erdős in [4] and proved in [6], Theorem 1:

Let $K_r^+(s_1, \dots, s_r)$ be the complete r -partite graph with parts of size $s_1 \geq 2, s_2, \dots, s_r$ with an edge added to the first part. For all $s \geq 2$, every graph of sufficiently large order n with $t_r(n) + 1$ edges contains a $K_r^+(s, \dots, s)$.

For $r = 2$, Erdős [4] gave a stronger result:

For all sufficiently small $\varepsilon > 0$, every graph of sufficiently large order n with $t_2(n) + 1$ edges contains a $K_2^+(\lfloor c \ln n \rfloor, \lceil n^{1-\varepsilon} \rceil)$ for some $c > 0$, independent of n .

We extend these two results as follows.

Theorem 1 *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(r+7)(r+1)}$, and G be a graph of order n . If G has $t_r(n) + 1$ edges, then G contains a $K_r^+(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$.*

Here is a simpler version of this assertion.

Theorem 2 *Let $r \geq 2$, $c = r^{-(r+7)(r+1)}$, $n \geq e^{2/c}$, and G be a graph of order n . If G has $t_r(n) + 1$ edges, then G contains a $K_r^+(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor)$.*

Theorems 1 and 2 have corresponding stability results.

Theorem 3 *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(r+7)(r+1)}/2$, $0 < \alpha < r^{-8}/8$, and G be a graph of order n . If G has $\lceil (1 - 1/r - \alpha)n^2/2 \rceil$ edges, then G satisfies one of the conditions:*

- (i) *G contains a $K_r^+(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-2\sqrt{c}} \rceil)$;*
- (ii) *G contains an induced r -partite subgraph G_0 of order at least $(1 - \sqrt{2\alpha})n$ and with minimum degree $\delta(G_0) > (1 - 1/r - 2\sqrt{2\alpha})n$.*

Theorem 4 *Let $r \geq 2$, $c = r^{-(r+7)(r+1)}/2$, $0 < \alpha < r^{-8}/8$, $n \geq e^{2/c}$, and G be a graph of order n . If G has $\lceil (1 - 1/r - \alpha)n^2/2 \rceil$ edges, then G satisfies one of the conditions:*

- (i) *G contains a $K_r^+(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor)$;*
- (ii) *G contains an induced r -partite subgraph G_0 of order at least $(1 - \sqrt{2\alpha})n$ and with minimum degree $\delta(G_0) > (1 - 1/r - 2\sqrt{2\alpha})n$.*

Remarks

- The relations between c and n in Theorems 1 and 3 need some explanation. First, for fixed c , they show how large must be n to get valid conclusions. But, in fact, the relations are subtler, for c itself may depend on n , e.g., letting $c = 1/\ln \ln n$, the conclusions are meaningful for sufficiently large n .
- Note that, in Theorems 1 and 3, if the conclusion holds for some c , it holds also for $0 < c' < c$, provided n is sufficiently large.
- The stability conditions in Theorems 3 and 4 are stronger than the conditions in the stability theorems of [5], [11] and [8]. Indeed, condition (ii) implies that G_0 is an induced, almost balanced, and almost complete r -partite graph containing almost all the vertices of G ;
- The exponents $1 - \sqrt{c}$ and $1 - 2\sqrt{c}$ in Theorems 1 and 3 are far from the best ones, but are simple.

In our proofs we apply the following two technical statements, which may be useful elsewhere.

Lemma 5 *Let $0 < \alpha \leq 1$, $1 \leq c \ln n \leq \alpha m/2 + 1$, and let F be a bipartite graph with parts A and B of size m and n . If $e(F) \geq \alpha mn$, then F contains a $K_2(s, t)$ with parts $S \subset A$ and $T \subset B$ such that $|S| = \lfloor c \ln n \rfloor$ and $|T| = t > n^{1-c \ln \alpha/2}$.*

Theorem 6 *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(r+8)r}$ and G be a graph G of order n . If G contains a K_{r+1} and has minimum degree $\delta(G) > (1 - 1/r - 1/r^4)n$, then G contains a*

$$K_r^+ \left(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \left\lceil n^{1-cr^3} \right\rceil \right).$$

The next section contains notation and results needed to prove the theorems. The proofs are presented in Section 3.

2 Preliminary results

Our notation follows [2]; given a graph G , we write:

- $V(G)$ for the vertex set of G and $|G|$ for $|V(G)|$;
- $E(G)$ for the edge set of G and $e(G)$ for $|E(G)|$;
- $\Gamma(u)$ for the set of neighbors of a vertex u and $d(u)$ for $|\Gamma(u)|$;
- $\delta(G)$ for the minimum degree of G ;
- $G[U]$ for the subgraph of G induced by a set $U \subset V(G)$;
- $H + u$ for $G[V(H) \cup \{u\}]$, where $H \subset G$ is a subgraph and $u \in V(G)$;
- $K_r(G)$ for the set of r -cliques of G and $k_r(G)$ for $|K_r(G)|$;
- $K_s(M)$ for the set of s -cliques contained in members of a set $M \subset K_r(G)$;
- $K_r(s_1, \dots, s_r)$ for the complete r -partite graph with parts of size s_1, \dots, s_r .

An r -joint of size t is the union of t distinct r -cliques sharing an edge. Write $js_r(G)$ for the maximum size of an r -joint in G .

Given a set $M \subset K_r(G)$, we say that M covers a subgraph $H \subset G$, if $E(H) \subset K_2(M)$.

The following facts play crucial roles in our proofs.

Fact 7 ([3], Lemma 1) *Let $r \geq 2$ and $c \geq 0$, and G be a graph of order n . If*

$$e(G) > (1 - 1/r + c)n^2/2,$$

then

$$k_{r+1}(G) > c \frac{r^2}{r+1} \left(\frac{n}{r}\right)^{r+1}.$$

□

Fact 8 ([3], Lemma 6) *Let $r \geq 2$, and G be a graph of order n . If G contains a K_{r+1} and $\delta(G) > (1 - 1/r - 1/r^4)n$, then $js_{r+1}(G) > n^{r-1}/r^{r+3}$.* □

Fact 9 ([3], Theorem 7) *Let $r \geq 2$, $n > r^8$, and G be a graph of order n . If $e(G) > t_r(n)$, then G has an induced subgraph G' of order $n' > (1 - 1/r^2)n$ such that either*

$$e(G') > \left(\frac{r-1}{2r} + \frac{1}{r^4(r^2-1)} \right) (n')^2 \tag{1}$$

or

$$K_{r+1} \subset G', \quad \text{and} \quad \delta(G') > (1 - 1/r - 1/r^4)n'. \tag{2}$$

□

Fact 10 ([7], Theorem 1) *Let $r \geq 2$, $\alpha^r \ln n \geq 1$, and G be a graph of order n . Every set $M \subset K_r(G)$ satisfying $|M| \geq \alpha n^r$ covers a $K_r(s, \dots, s, t)$ with $s = \lfloor \alpha^r \ln n \rfloor$ and $t > n^{1-\alpha^{r-1}}$. \square*

3 Proofs

Proof of Lemma 5 Set $s = \lfloor c \ln n \rfloor$ and let

$$t = \max \{x : \text{there exists } K_2(s, x) \subset F \text{ with part of size } s \text{ in } A\}.$$

Thus $d(X) \leq t$ for each $X \subset A$ with $|X| = s$, and so,

$$t \binom{m}{s} \geq \sum_{X \subset A, |X|=s} d(X) = \sum_{u \in B} \binom{d(u)}{s}. \quad (3)$$

Setting

$$f(x) = \begin{cases} \binom{x}{s} & \text{if } x \geq s-1 \\ 0 & \text{if } x < s-1, \end{cases}$$

and noting that $f(x)$ is a convex function, we find that,

$$\sum_{u \in B} \binom{d(u)}{s} = \sum_{u \in B} f(d(u)) \geq n f\left(\frac{1}{n} \sum_{u \in B} d(u)\right) = n \binom{e(F)/n}{s} \geq n \binom{\alpha m}{s}.$$

Combining this inequality with (3) and rearranging, we find that

$$t \geq n \frac{\alpha m (cm-1) \cdots (\alpha m - s + 1)}{m(m-1) \cdots (m-s+1)} > n \left(\frac{\alpha m - s + 1}{m} \right)^s \geq n \left(\frac{\alpha}{2} \right)^s \geq n^{1+c \ln(\alpha/2)},$$

completing the proof. \square

Proof of Theorem 6 Let r, c, n , and the graph G satisfy the conditions of the theorem. Note first that for every $R \in K_{r-1}(G)$,

$$\begin{aligned} d(R) &= \left| \bigcap_{u \in R} \Gamma(u) \right| \geq \sum_{u \in R} d(u) - (r-2)n \geq (r-1)\delta(G) - (r-2)n \\ &> \left((r-1) \left(\left(1 - \frac{1}{r} \right) - \frac{1}{r^4} \right) - (r-2) \right) n > \frac{n}{r^2}. \end{aligned} \quad (4)$$

Also, Fact 8 implies that

$$js_{r+1}(G) > \frac{n^{r-1}}{r^{r+3}} > \left(1 - \frac{1}{r^2} \right)^{r-1} \frac{n^{r-1}}{r^{r+3}} > \left(1 - \frac{r-1}{r^2} \right) \frac{n^{r-1}}{r^{r+3}} > \frac{n^{r-1}}{r^{r+4}}.$$

Thus, there exists an edge $uv \in E(G)$ contained in more than n^{r-1}/r^{r+4} distinct $(r+1)$ -cliques of G . Letting $B = \Gamma(u) \cap \Gamma(v) \cap V(G)$, we see that

$$k_{r-1}(G[B]) > n^{r-1}/r^{r+4}. \quad (5)$$

Define the set X as

$$X = \{R : R \in K_r(G) \text{ and } |R \cap B| \geq r-1\}.$$

In view of (4) and (5), we find that

$$|X| \geq \frac{1}{r} \sum_{P \in K_{r-1}(G[B])} d(P) > \frac{1}{r} \times \frac{n}{r^2} \times \frac{n^{r-1}}{r^{r+4}} = \frac{n^r}{r^{r+7}},$$

For a set $N \subset K_r(G)$ and a clique $R \in K_{r-1}(N)$ let $d_N(R)$ be the number of members of N containing R . We claim that there exists $Y \subset X$ with $|Y| > n^r/r^{r+8}$ such that $d_Y(R) > n/r^{r+8}$ for all $R \in K_{r-1}(Y)$. Indeed, set $Y = X$ and apply the following procedure:

While *there exists an $R \in K_{r-1}(Y)$ with $d_Y(R) \leq n/r^{r+8}$* **do**
 Remove from Y all r -cliques containing R .

When the procedure stops, $d_Y(R) > n/r^{r+8}$ for all $R \in K_{r-1}(Y)$, and

$$|X| - |Y| \leq |K_{r-1}(X)| \frac{n}{r^{r+8}} \leq \binom{n}{r-1} \frac{n}{r^{r+8}} < \frac{1}{r^{r+8}} n^r,$$

implying that $|Y| > n^r/r^{r+8}$, as claimed.

Since

$$|K_{r-1}(Y)| \geq r|Y|/n > r \times n^{r-1}/r^{r+8} = n^{r-1}/r^{r+7},$$

by Fact 10, $K_{r-1}(Y)$ covers a subgraph $H = K_{r-1}(m, \dots, m)$ with $m = \lfloor r^{-(r+7)(r-1)} \ln n \rfloor$.

Select a set A of m disjoint $(r-1)$ -cliques in H and define a bipartite graph F with parts A and B , joining $R \in A$ to $v \in B$ if $R + v \in Y$.

Let $\alpha = 1/r^{r+8}$ and set $s = \lfloor c \ln n \rfloor$. Since

$$d_Y(R) > \frac{1}{r^{r+8}} n \geq \alpha n$$

for all $R \in K_{r-1}(Y)$, we have $e(F) > \alpha mn$. Also, we find that

$$s \leq c \ln n \leq \frac{1}{r^{(r+8)r}} \ln n \leq \frac{1}{2r^{r+8}} \times \frac{1}{r^{(r+7)(r-1)}} \ln n \leq \frac{\alpha}{2} m + 1.$$

Hence, by Fact 5, H contains a $K_2(s, t)$ with parts $S \subset A$ and $T \subset B$ such that $|S| = s$ and $|T| = t > n^{1-c \ln \alpha/2}$. A routine calculation shows that for $r \geq 2$,

$$\ln \alpha/2 = \ln \frac{1}{2r^{r+8}} \geq -r^3,$$

and so, $t > n^{1-cr^3}$.

Letting H^* be the subgraph of H induced by the union of the members of S , we see that $H^* = K_{r-1}(s, \dots, s)$. Note that at least $(r-2)$ of the parts of H^* belong to B , for otherwise we can select an $(r-1)$ -clique Q in H^* with two vertices outside B , and so, every $R \in Y$ containing

Q has two vertices outside B . This is a contradiction since $Y \subset X$ and all members of X intersect B in at least $r - 1$ vertices.

Let H_1, \dots, H_{r-1} be the parts of H^* , and assume by symmetry that $H_i \subset B$ for $i = 2, \dots, r - 1$. Remove two vertices from H_1 , add u and v to H_1 , and write H'_1 for the resulting set. Clearly the sets $H'_1, H_2, \dots, H_{r-1}, T$ induce a subgraph containing a $K_r^+ \left(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-cr^3} \rceil \right)$, completing the proof. \square

Proof of Theorem 1 Let G be a graph of order n with $t_r(n) + 1$ edges. Fact 9 implies that there exists an induced subgraph $G' \subset G$ of order $n' > (1 - 1/r^2)n$ such that either (1) or (2) holds.

Assume first that G' satisfies condition (1). Fact 7 implies that

$$\begin{aligned} k_{r+1}(G) &\geq k_{r+1}(G') > \frac{2}{r^4(r^2 - 1)} \times \frac{r^2}{r + 1} \times \left(\frac{n'}{r}\right)^{r+1} \\ &> \frac{2}{r^2(r^2 - 1)(r + 1)} \times \left(1 - \frac{1}{r^2}\right)^{r+1} \times \left(\frac{n}{r}\right)^{r+1} \\ &> \frac{2}{r^2(r^2 - 1)(r + 1)} \times \left(1 - \frac{r + 1}{r^2}\right) \times \left(\frac{n}{r}\right)^{r+1} \\ &> \frac{2(r^2 - r - 1)}{r^4(r^2 - 1)(r + 1)} \times \left(\frac{n}{r}\right)^{r+1} > \frac{1}{r^{r+7}} n^{r+1} > c^{1/(r+1)} n^{r+1}. \end{aligned}$$

Hence, by Fact 10, G contains a $K_{r+1}(s, \dots, s, t)$ with $s = \lfloor c \ln n \rfloor$ and

$$t > n^{1-c^{r/(r+1)}} > n^{1-\sqrt{c}}.$$

Then, obviously, G contains a $K_r^+(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$, completing the proof.

Finally, assume that G' satisfies condition (2). Applying Theorem 6, we see that G' contains a

$$K_r^+(\lfloor 2c \ln n' \rfloor, \dots, \lfloor 2c \ln n' \rfloor, \lceil (n')^{1-2cr^3} \rceil).$$

To complete the proof, note that

$$2c \ln n' \geq 2c \ln \left(1 - \frac{1}{r^2}\right) n \geq 2 \ln \left(1 - \frac{1}{r^2}\right) + 2 \ln n \geq c \ln n$$

and

$$(n')^{1-2cr^3} \geq \left(1 - \frac{1}{r^2}\right)^{1-2cr^3} n^{1-2cr^3} \geq \left(1 - \frac{1}{r^2}\right) n^{1-2cr^3} > n^{1-\sqrt{c}}.$$

\square

Proof of Theorem 3 Let G be a graph of order n with $e(G) > (1 - 1/r - \alpha)n^2/2$. Set $V = V(G)$, $\varepsilon = \sqrt{2\alpha}$, and define the set M_ε as

$$M_\varepsilon = \{u \in V(G) : d(u) \leq (1 - 1/r - \varepsilon)n\}.$$

Assume that condition (i) fails. We shall show that: (a) $|M_\varepsilon| < \varepsilon n$; (b) the graph $G_0 = G[V \setminus M_\varepsilon]$ satisfies condition (ii).

(a) The set M_ε satisfies $|M_\varepsilon| < \varepsilon n$

Assume for a contradiction that $|M_\varepsilon| \geq \varepsilon n$, select $M' \subset M_\varepsilon$ with

$$|M'| = \lfloor \varepsilon n \rfloor \quad (6)$$

and note that M' is nonempty since $\varepsilon n = \sqrt{2\alpha}n > 1$. Letting $G' = G[V \setminus M']$, we see that

$$\begin{aligned} e(G) &= e(G') + e(M', V \setminus M') + e(M') \leq e(G') + \sum_{u \in M'} d(u) \\ &\leq e(G') + |M'| (1 - 1/r - \varepsilon) n. \end{aligned}$$

Assume for a contradiction that

$$e(G') > \frac{r-1}{2r} (n - |M'|)^2$$

and set $p = n - |M'|$. In view of (6), we have

$$p \geq n - \varepsilon n = (1 - \sqrt{2\alpha}) n.$$

Hence, by Theorem 1, G contains a $K_r^+ \left(\lfloor 2c \ln p \rfloor, \dots, \lfloor 2c \ln p \rfloor, \left\lceil p^{1-\sqrt{2c}} \right\rceil \right)$. Since

$$2c \ln p \geq 2c \ln \left((1 - \sqrt{2\alpha}) n \right) \geq 2c \ln \left(1 - \frac{1}{4r^4} \right) n \geq c \ln n$$

and

$$p^{1-\sqrt{2c}} \geq (1 - \sqrt{2\alpha})^{1-\sqrt{2c}} n^{1-\sqrt{2c}} > (1 - \sqrt{2\alpha}) n^{1-\sqrt{2c}} > n^{1-2\sqrt{c}},$$

this contradicts the assumption that (i) fails.

Hereafter, we assume that

$$e(G') \leq \frac{r-1}{2r} (n - |M'|)^2.$$

From

$$e(G') \geq e(G) - \sum_{u \in M} d(u) \geq (1 - 1/r - \alpha) n^2/2 - |M'| (1 - 1/r - \varepsilon) n,$$

we obtain

$$\frac{r-1}{2r} (n - |M'|)^2 \geq \left(\frac{r-1}{r} - \alpha \right) \frac{n^2}{2} - |M'| \left(\frac{r-1}{r} - \varepsilon \right) n.$$

After some algebra, we find that

$$|M'| < \left(\varepsilon - \sqrt{\varepsilon^2 - \alpha} \right) n = \varepsilon \left(1 - \sqrt{1/2} \right) n$$

or

$$|M'| > \left(\varepsilon + \sqrt{\varepsilon^2 - \alpha} \right) n = \varepsilon \left(1 + \sqrt{1/2} \right) n,$$

contradicting (6) in view of $\varepsilon\sqrt{1/2}n = \sqrt{2\alpha}n > \sqrt{2}$. Therefore, $|M_\varepsilon| < \varepsilon n$.

(b) The graph $G_0 = G[V \setminus M_\varepsilon]$ satisfies condition (ii).

By our choice of M_ε , for every $u \in V \setminus M_\varepsilon$, we have $d(u) > (1 - 1/r - \varepsilon)n$; thus

$$\delta(G_0) > (1 - 1/r - \varepsilon)n - |M_\varepsilon| > (1 - 1/r - 2\varepsilon)n = \left(1 - 1/r - 2\sqrt{2\alpha} \right) n,$$

and so, $\delta(G_0)$ satisfies the required condition. All that remains to prove is that G_0 is r -partite.

If G_0 contains a K_{r+1} , in view of

$$\delta(G_0) > \left(1 - 1/r - 2\sqrt{2\alpha} \right) n > \left(1 - 1/r - 1/r^4 \right) n,$$

using Theorem 6 as in the proof of Theorem 1, we see that G contains a

$$K_r^+ \left(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \left\lceil n^{1-\sqrt{c}} \right\rceil \right),$$

contradicting our assumption. Thus, G_0 is K_{r+1} -free. In view of

$$\delta(G_0) > \left(1 - 1/r - 1/r^4 \right) n > \left(1 - \frac{3}{3r-1} \right) |G_0|,$$

the theorem of Andrásfai, Erdős and Sós [1] implies that G_0 is r -partite, completing the proof. \square

We omit the proofs of Theorems 2 and 4, since they are easy consequences of Theorem 1 and 3.

Concluding remark

Finally, a word about the project mentioned in the introduction: in this project we aim to give wide-range results that can be used further, adding more integrity to extremal graph theory.

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